PRELIMINARY

On the Wave Propagation in Nonuniform Media*

The scattering of plane waves by nonuniform media is currently of considerable interest.[1,2,3] Models of this type considered by many authors are confined to simple cases where the permittivity ϵ is a function only of one dependent variable, and the permeability μ, is constant. Problems dealt with in several cases are actually second order homogeneous differential equations. The purpose of this paper is to report that the method of collocation[4] is applicable to achieve an approximate solution. This method has two advantages: (1) There is no limitation on the variation of ϵ as long as it is a well-behaved function. (2) Good solutions can be achieved even when values of ϵ are known (by experiment) only at a sufficient number of points in space.

Consider the propagation of a plane wave through an infinite, nonuniform dielectric slab which was located in the region $0 \leq x \leq a$. (x is a Cartesian coordinate). The permittivity ϵ is a real regular function of x only. The differential equation involved in this case is of the form[1]

$$U''(x) + p(x)\,U'(x) + q(x)\,U(x) = 0, \qquad (1)$$

where the prime denotes the derivative with respect to argument, $p(x)$ and $q(x)$ are in general regular functions of both $\epsilon(x)$ and $\epsilon'(x)$. Two independent particular solutions of Eq. (1) may be, approximately, expressed by

---

* This work was supported by NASA under Grant NsG 608.

1 J. H. Richmond, "Transmission Through Inhomogeneous Plane Layers," IRE Trans. on Antenna and Propagation, vol. AP-10, p. 300, May 1962.

2 F. A. Albini and E. R. Nagelberg, "Scattering of a Plane Wave by an Infinite Inhomogeneous Dielectric Cylinder - An Application of the Born Approximation," J. Appl. Phys. p. 1706, vol. 33, May 1962.

3 Donald Amush, "Electromagnetic Scattering from a Spherical Nonuniform Medium-Part 1 General Theory", IEEE Trans. on Antennas and Propagation, vol. AP-12, p. 87, January, 1964.

4 F. B. Hildebrand, "Method of Applied Mathematics," Prentice Hall, Inc., Englewood Cliffs, N. J., pp. 452-459, 1954.

$$U_e(x) = \sum_{n=0}^{N} A_n \cos L_{1n} x\,, \qquad \text{even function} \tag{2}$$

$$U_o(x) = \sum_{m=1}^{M} B_m \sin L_{2m} x\,, \qquad \text{odd function} \tag{3}$$

where $L_{1n} = n\pi/v_1 a$, $L_{2m} = m\pi/v_2 a$, M and N are integers. The dimensionless quantities, v_1 and v_2, to be determined by the differential equation, are two real numbers greater than or equal to unity. They are nearly equal if N and M approach to infinity.

The odd solution only is considered here. The even solution can be obtained by the same procedure. Substituting Eq. (3) into Eq. (1) yields

$$\sum_{m=1}^{M} \left\{ [q(x) - L_{2m}^2] \sin L_{2m} x + p(x) L_{2m} \cos L_{2m} x \right\} B_m = 0. \tag{4}$$

Eq. (4) must satisfy all points within the region $0 \leq x \leq a$. But for the purpose of approximation, the method of collocation requires the equality to be fulfilled only at M points. Let these points be $0 \leq x_1 < x_2 < \ldots < x_m = a$. There are many choices of points. Usually, it is convenient to choose equal spaces between points. For each point, Eq. (4) is an algebraic equation of M unknowns, B_m. Hence, a system of M algebraic equations with M unknowns is then formed, the rest of the work is devoted to solve an eigenvalue and eigenvector problem. That is

$$[D_{im}] \; [B_m] = 0\,, \tag{5}$$

where $D_{im} = [q(x_i) - L_{2m}^2] \sin L_{2m} x_i + p(x_i) L_{2m} \cos L_{2m} x_i$. The value of v_2 is determined by

$$\det \left| D_{im} \right| = 0. \tag{6}$$

There are many roots of v_2 in Eq. (6). Taking the covergence into account, the suitable value is the smallest root which is greater than or equal to unity. With the known value of v_2 the expansion coefficients B_m can be calculated from Eq. (5) in terms of a B_r, which is the largest among the B_m's. By this method the approximate general solution of Eq. (1) is found for a specific frequency within the region $0 \leq x \leq a$. It should be mentioned that the method of least squares is applicable too.[4,5]

The partial wave analysis of scattering a plane wave by a cylindrically symmetric nonuniform dielectric cylinder, or by a spherically symmetric nonuniform dielectric sphere, would result one (or two) differential equation of the following form[3,5]

$$L_n V_n (r) + p(r) V'_n (r) + [q(r) - 1] V_n (r) = 0, \tag{7}$$

where $p(r)$ and $q(r)$ are regular functions of r within the region $0 \leq r \leq a$. The operator L_n is defined as $L_n Z_n(r) = 0$, if $Z_n(r)$, is the Bessel function of n^{th} order for the cylindrical case, $Z_n(r)$ is the spherical Bessel function of n^{th} order for the spherical case. Analogous to Eqs. (2) and (3) of the plane case, the regular particular solution of Eq. (7) may be approximated by

$$V(r) = \sum_{m=1}^{M} C_m Z_n (\alpha_m \xi), \tag{8}$$

where $\xi = r/ua$, $Z_n(\alpha_m) = 0$, and Z_n is the first kind Bessel function. The subscript n of $V_n(r)$, C_{nm} and α_{nm} are omitted. The dimensionless parameter, u, to be determinated by the differential equation, is a real number greater than or equal to unity. Substituting Eq. (8) into (7) yields

$$\sum_{m=1}^{M} \left\{ \frac{\alpha_m}{ua} p(r) Z'_n (\alpha_m \xi) + \left[q(r) - \left(\frac{\alpha_m}{ua}\right)^2 \right] Z_n (\alpha_m \xi) \right\} C_m = 0. \tag{9}$$

Note that Eq. (9) is similar to Eq. (4). The same procedures may be used to determine the suitable value of u and the expansion coefficients, C_m. In other words, if all the sine functions are replaced by Z_n, cosine functions by Z'_n,

---

5 H. Y. Yee, "Approximate Methods for the Computation of Wave Propagation in Nonuniform Media," Wave Propagation Interim Report No. 1, University of Alabama Research Institute, Huntsville, Alabama, September 1964.

$\frac{m\pi}{v_2}$ by $\frac{\alpha_m}{u}$, and B_m by C_m in Eqs. (4) - (6), then all these equations and the associated statements are valid for the solution of Eq. (7).

Hung Yuet Yee
University of Alabama Research Institute
Huntsville, Alabama

Frederick J. Tischer
Department of Electrical Engineering
North Carolina State College
Raleigh, North Carolina